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Approximations with Negative Roots and Poles

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We ask when best uniform rational or polynomial approximations on $[0, 1]$ have negative roots or poles. We show that the best $(n+k, n)$th rational approximation to a Stieltjes transform has only negative poles. We use this to show that the best $(n+k, n)$th rational approximation is better than the best $(2n+k-1)$th polynomial approximation to such functions. We also construct a class of entire functions whose best polynomial approximations have negative roots. We show that for this class the best $(n+m+1)$th polynomial approximation behaves better than the best (n, m)th rational approximation.

Let π_n denote the collection of polynomials of degree at most n with real coefficients ($\pi_{-1} \equiv 0$). If f is continuous on $[a, b]$ we set

$$E_n(f\colon [a, b]) = \min_{p_n \in \pi_n} \|f - p_n\|_{[a,b]}$$

and

$$R_{n,m}(f\colon [a, b]) = \min_{p_n \in \pi_n, q_m \in \pi_m} \|f - p_n/q_m\|_{[a,b]},$$

where $\|\cdot\|_{[a,b]}$ denotes the supremum norm on $[a, b]$. When we talk about best approximations it will be in this norm.

We prove the following

THEOREM 1. *Let $r_k(x) \in \pi_k$, let α be non-decreasing and let*

$$f(x) = \int_0^\infty \frac{1}{x+t}\, d\alpha(t) + r_k(x). \tag{1}$$

Suppose f is defined (as a convergent Stieltjes integral) for $x \geqslant c \geqslant 0$. Suppose that $p_{n+k} \in \pi_{n+k} (k \geqslant -1)$, $q_n \in \pi_n$ and suppose that

$$p_{n+k}(x) - q_n(x)\, f(x)$$

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has $2n+k+1$ *zeroes on* $[a, b]$, $a \geqslant c$. *Then* q_n *has all negative roots and*

$$\frac{p_{n+k}(x)}{q_n(x)} = s_k(x) + \sum_{i=1}^{n} \frac{\gamma_i}{x+\delta_i}, \tag{2}$$

where $\gamma_i, \delta_i > 0$ *for all* i *and where* $s_k \in \pi_k$.

A function of the form $\int_0^\infty 1/(x+t)\, d\alpha(t)$ is called a Stieltjes transform (of α). In the context of this paper Stieltjes transforms will always be of non-decreasing α. We have the following characterization of Stieltjes transforms.

COROLLARY 1. *Fix* $k \geqslant -1$. *The following two conditions are equivalent for a non-rational* f.

(A) f *can be represented as*

$$f(x) = \int_0^\infty \frac{1}{x+t}\, d\alpha(t) + r_k(x),$$

where r_k *is a polynomial of degree* $\leqslant k$, α *is a non-decreasing function and the above Stieltjes integral converges for* $x \geqslant a \geqslant 0$.

(B) f *is continuous on* $[a, b]$, *for some* $b > a \geqslant 0$ *and for all* n *the best* $(n+k, n)$ *rationl approximation to* f *on* $[a, b]$ *is of the form*

$$s_k^n(x) + \sum_{i=1}^{n} \frac{\gamma_i^n}{x+\delta_i^n},$$

where $\gamma_i^n, \delta_i^n > 0$ *and* s_k^n *is a polynomial of degree* $\leqslant k$.

Two cases of Theorem 1 are known. The case $k = -1$ is proved by Kreın [3, p. 166; or 5, p. 96]. The case where p_{n+k}/q_n is the Padé approximant (that is, all the interpolation points are the same) is proved by Baker (see [1]).

We require the following lemmas.

LEMMA 1. *Consider, for* m *a positive integer,*

$$f(x) = \sum_{i=0}^{\infty} \frac{a_i}{(x+\gamma_i)^m},$$

where $\gamma_{i+1} \geqslant \gamma_i > 0$ *and each* a_i *is real. Then the number of zeroes of* f *on* $[0, \infty)$ *is no greater than the number of sign changes in the sequence* $\{a_0, a_1, a_2, ...\}$ *(*a_i *terms that vanish are ignored and zeroes are counted according to their multiplicities).*

LEMMA. 2. *Suppose that α is non-decreasing. Consider, for a positive integer m and a polynomial g,*

$$f(x)=\int_0^\infty \frac{g(t)}{(x+t)^m}\,d\alpha(t).$$

If f has k possibly multiple non-negative zeroes then g has at least k distinct positive zeroes.

Both lemmas are immediate consequences of results in [3] or [4]. The basic point in Lemma 1 is that

$$(-1)^m m! \sum_{i=1}^{n} \frac{a_i}{(x+\gamma_i)^{m+1}} = \int_0^\infty (-1)^m \left(\sum_{i=1}^{n} a_i e^{-\gamma_i t}\right) e^{-xt}\,dt.$$

Lemma 1 now follows from the variation diminishing properties of the Laplace transform and Descartes rule of signs. Lemma 2 can be deduced from Lemma 1 by approximating α by step functions.

We note that Lemma 1 implies that

$$\left\{\frac{1}{(x+\alpha_1)^k},\ldots,\frac{1}{(x+\alpha_n)^k}\right\}, \qquad \alpha_j \neq \alpha_i > 0,$$

is a Tchebycheff system on any positive interval.

Proof of Theorem 1. Let $q_n(x)=\sum_{k=0}^{n} a_k x^k$. Let $h=n+k+1$. Consider

$$(q_n(x)\cdot f(x))^{(h)} = \sum_{m=0}^{h} \binom{h}{m} q_n^{(m)}(x)\, f^{(h-m)}(x).$$

We note that $(q_n(x)\cdot f(x))^{(h)}$ has n zeroes on $[a,b]$. We also note that we do not need to assume that the zeroes of $p_{n+k}-q_n f$ are distinct. Since $q_n^{(m)}\equiv 0$ for $m>n$ we have, for $x \geqslant c$,

$$\begin{aligned}
(q_n(x)\cdot f(x))^{(h)} &= \sum_{m=0}^{n} \binom{h}{m} q_n^{(m)}(x)\, f^{(h-m)}(x)\\
&= \sum_{m=0}^{n} \binom{n}{m} \left(\sum_{k=m}^{n} \frac{a_k x^{k-m} k!}{(k-m)!}\right)\left(\int_0^\infty \frac{(h-m)!(-1)^{h-m}}{(x+t)^{h+1-m}}\,d\alpha(t)\right)\\
&= \sum_{k=0}^{n} \sum_{m=0}^{k} (-1)^h h!\, a_k x^k \int_0^\infty \frac{k!(-x)^{-m}}{(k-m)!\, m!(x+t)^{h+1-m}}\,d\alpha(t)\\
&= \sum_{k=0}^{n} (-1)^h h!\, a_k x^k \int_0^\infty \left(\sum_{m=0}^{k} \binom{k}{m} \left(\frac{-x}{x+t}\right)^{-m}\right) \frac{d\alpha(t)}{(x+t)^{h+1}}
\end{aligned}$$

$$= \sum_{k=0}^{n} (-1)^h h!\, a_k x^k \int_0^\infty (-1)^k \left(\frac{t}{x}\right)^k \frac{d\alpha(t)}{(x+t)^{h+1}}$$

$$= (-1)^h h! \int_0^\infty \left(\sum_{k=0}^{n} a_k(-t)^k\right) \frac{d\alpha(t)}{(x+t)^{h+1}}$$

$$= (-1)^h h! \int_0^\infty \frac{q_n(-t)}{(x+t)^{h+1}}\, d\alpha(t). \tag{3}$$

We observe, by Lemma 2, that

$$\int_0^\infty \frac{q_n(-t)}{(x+t)^{h+1}}\, d\alpha(t)$$

can have n non-negative zeroes only if $q_n(-t)$ has n positive zeroes. It follows that q_n has only negative roots and that these roots are distinct. Thus,

$$\frac{p_{n+k}(x)}{q_n(x)} = s_k(x) + \sum_{i=1}^{n} \frac{b_i}{x+\delta_i},$$

where $s_k \in \pi_k$, $\delta_i > 0$. It remains to show that $b_i > 0$. If, for $x \geqslant c$,

$$f(x) - r_k(x) = \int_0^\infty \frac{1}{x+t}\, d\alpha(t)$$

then there exists m, β_i, $\eta_i \geqslant 0$ so that

$$\left\| \sum_{i=1}^{m} \frac{\beta_i}{x+\eta_i} - f(x) + r_k(x) \right\|_{[c,\infty)} < \varepsilon.$$

If we consider $(k+1)$st derivatives we see that, for an appropriate ε,

$$\sum_{i=1}^{n} \frac{b_i}{(x+\delta_i)^{k+2}} - \sum_{i=1}^{m} \frac{\beta_i}{(x+\eta_i)^{k+2}}$$

has $2n$ zeroes on $[c, \infty)$ and we deduce from Lemma 1 that each $b_i > 0$. ∎

Proof of Corollary 1. That condition A implies condition B follows from Theorem 1 and the alternation criteria for best rational approximations (see [2, p. 158]). Calculation (3) of the proof of Theorem 1 allows us to deduce that if f is not a rational function then the best $(n+k, n)$ rational approximation to f is nondegenerate (see [6, pp. 163–165]).

The proof that B implies A is a consequence of results in Widder (see

[10, p. 364]). We observe that since π_k is finite dimensional we have, passing to subsequences if necessary,

$$\lim_{n\to\infty} s_k^n(x) = r_k(x).$$

Also,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\gamma_i^n}{x+\delta_i^n} = \int_0^\infty \frac{1}{x+t}\, d\alpha(t).$$

Each rational function of the form

$$\sum_{i=1}^{n} \frac{\gamma_i^n}{x+\delta_i^n}$$

is a Stieltjes transform of an increasing (step) function β_n. It is now possible, via Helly's theorem, to write α as a limit of the β_n on $[a, \infty)$. ∎

One can deduce from Corollary 1 and Lemma 1 that the poles of the best $(n+k, n)$th rational approximation to a Stieltjes transform interlace with the poles of the best $(n+k+1, n+1)$th approximation.

THEOREM 2. *Suppose f is continuous on $[a, b], a \geqslant 0$ and suppose the best $(n+k, n)$ rational approximation to f on $[a, b]$ is of the form*

$$s_k(x) + \sum_{i=1}^{n} \frac{\gamma_i}{1+\delta_i x},$$

where $s_k \in \pi_k$ and $\gamma_i, \delta_i > 0$. Then

$$R_{n+k,n}(f\colon [a, b]) \leqslant P_{2n+k-1}(f\colon [a, b]).$$

Proof. Let

$$r(x) = s_k(x) + \sum_{i=1}^{n} \frac{\gamma_i}{1+\delta_i x}$$

be the best $(n+k, n)$th rational approximation to f on $[a, b]$ and let $p(x)$ be the best $(2n+k-1)$th polynomial approximation to f on $[a, b]$. We assume that

$$P_{2n+k-1}(f\colon [a, b]) < R_{n+k,n}(f\colon [a, b])$$

and derive a contradiction. Under the above assumption, appealing to the usual alternation criteria for best approximation, we deduce that $r(x)-p(x)$ has $2n+k+1$ zeroes on $[a,b]$ and hence that

$$(r(x)-p(x))^{(2n+k)}=(r(x))^{(2n+k)}$$

has a zero on $[a,b]$. This is impossible since

$$(r(x))^{2n+k}=(-1)^{2n+k}(2n+k)!\sum_{i=1}^{n}\frac{(\delta_i)^{2n+k}\gamma_i}{(1+\delta_i x)^{2n+k+1}}$$

is never zero on $[a,b]$. ∎

The above theorem can be applied, by Corollary 1, to Stieltjes transforms. We note that $\log(x+1)/x$ and $x^{-\delta}$, $0<\delta<1$, are Stieltjes transforms [10, p. 346].

POLYNOMIAL APPROXIMATIONS WITH REAL ROOTS

Let Γ be the set of entire functions defined by

$$\Gamma=\left\{\sum_{n=0}^{\infty}a_n z^n \,\middle|\, 0\leqslant 4\cdot a_{n+1}\leqslant (a_n)^2 \text{ and } a_0<\tfrac{1}{3}\right\}$$

and

$$\Gamma^*=\left\{\sum_{n=0}^{\infty}a_n z^n\in\Gamma \,\middle|\, a_0\leqslant 5/16,\ a_1\leqslant a_0^2/10 \text{ and } a_2\leqslant a_1^2/5\right\}.$$

THEOREM 3. (a) *If $f\in\Gamma$, then for all n the* nth *partial sum of f has only negative roots.*

(b) *If $f\in\Gamma^*$ then every best uniform polynomial approximation to f on* $[-1,1]$ *has only negative roots.*

Polya and Szegö [8, p. 66] show that

$$f(z)=\sum_{n=0}^{\infty}\frac{z^n}{a^{n^2}},\qquad a>2,$$

has the property that all its partial sums have negative roots. We use an analogous argument for Theorem 3.

Proof. To prove (a) we suppose that

$$f(z)=\sum_{n=0}^{\infty}a_n z^n,\qquad 0<4a_{n+1}\leqslant(a_n)^2,\qquad a_0<\tfrac{1}{3}$$

and consider

$$S_N(z) = \sum_{n=0}^{N} a_n z^n.$$

We evaluate

$$\frac{S_N(-1/a_n)}{a_n(-1/a_n)^n}.$$

For $j > 0$

$$\left| \frac{a_{n+j}(-1/a_n)^{n+j}}{a_n(-1/a_n)^n} \right| \leqslant \frac{a_{n+j}}{(a_n)^{j+1}} \leqslant \frac{1}{4^j}.$$

For $j = -n$

$$\left| \frac{a_0}{a_n(-1/a_n)^n} \right| \leqslant a_0 < \tfrac{1}{3}.$$

For $-n < j \leqslant -1$,

$$\left| \frac{a_{n+j}(-1/a_n)^{n+j}}{a_n(-1/a_n)^n} \right| \leqslant a_{n+j} \leqslant \frac{1}{4^{n+j}}.$$

Therefore,

$$\frac{S_N(-1/a_n)}{a_n(-1/a_n)^n} \geqslant 1 - \frac{1}{3} - 2\sum_{k=1}^{n} \frac{1}{4^j} > 0.$$

Thus, S_N changes sign between $-1/a_n$ and $-1/a_{n+1}$ and, hence, has real negative roots.

We now prove part (b). We need the following inequality (see [9, p. 226]): If $p_n \in \pi_n$ then

$$|p_n^{(k)}(0)| \leqslant n^k \, \|p_n\|_{[-1,1]}.$$

Suppose $p_n = \sum_{h=0}^{n} b_h x^h$ is the best nth degree polynomial approximation to f on $[-1, 1]$. Then

$$\|p_n - s_n\|_{[-1,1]} \leqslant 2 \sum_{h=n+1}^{\infty} a_h \leqslant 4a_{n+1}$$

and

$$|p_n^{(k)}(0) - s_n^{(k)}(0)| \leqslant 4n^k a_{n+1}.$$

Thus,

$$|b_k - a_k| \leqslant (4n^k/k!)\, a_{n+1}.$$

Since

$$a_{n+1} \leqslant \frac{(a_k)^{2(n+1-k)}}{4^{(n+1-k)}} \qquad \text{and} \qquad a_k \leqslant \left(\frac{1}{4}\right)^{2^k}$$

we have, for all $0 < k \leqslant n$,

$$|b_k - a_k| \leqslant \frac{n^k a_k}{k!\, 4^{2k(2^{n+1-k}-1)}} \leqslant \frac{a_k}{8}.$$

It follows that, for $0 < k \leqslant n$,

$$0 \leqslant b_{k+1} \leqslant (9/8)\, a_{k+1} \leqslant (9/32)\, a_k^2 \leqslant (18/49)\, b_k^2.$$

If we consider $q_n(x) = p_n(2x)$ we see that q_n satisfies the conditions of part (a) provided $a_0 \leqslant 5/16$, $a_1 \leqslant a_0^2/10$ and $288 a_2 \leqslant 49 a_1^2$. ∎

Contained in the Polya class is the set of functions which are uniform limits of polynomials with negative roots. These functions are all of the form

$$f(x) = \gamma e^{\alpha x} \prod_{i=1}^{\infty} (1 + \delta_i x), \qquad \alpha, \delta_i \geqslant 0.$$

Since neither all the partial sums nor all the best polynomial approximations to $e^{\alpha x}$ on $[0, 1]$ have all negative roots it is apparent that Γ is a proper subclass of this class (see [7]). The next theorem can be applied to the class Γ^*.

THEOREM 4. *Suppose that f is continuous on $[a, b]$, $a \geqslant 0$ and suppose that the best polynomial approximation of degree n to f has only negative roots. Then*

$$E_n(f\colon [a, b]) \leqslant R_{n-k,k-1}(f\colon [a, b]).$$

Proof. Let $p \in \pi_n$ be the best polynomial approximation to f on $[a, b]$. Let r/s be the best $(n-k, k-1)$th approximation to f on $[a, b]$ where $r \in \pi_{n-k}$, $s \in \pi_{k-1}$. Suppose, for the sake of a contradiction, that $R_{n-k,k-1}(f\colon [a, b]) < E_n(f\colon [a, b])$. Once again, appealing to the alternation characterization of best polynomial approximations, we deduce that

$$r(x) - s(x) \cdot p(x)$$

has $n+1$ zeroes on $[a, b]$. This implies that

$$(r(x) - s(x) \cdot p(x))^{(n+1-k)}$$

has at least k zeroes on $[a, b]$. However, since $p(x)$ has n zeroes on $(-\infty, 0)$,

$$(r(x) - s(x) \cdot p(x))^{(n+1-k)} = (p(x) \cdot s(x))^{(n+1-k)}$$

has $k-1$ zeroes on $(-\infty, 0)$. This yields the contradiction that the polynomial

$$(r(x) - s(x) \cdot p(x))^{(n+1-k)}$$

of degree $2k-2$ has $2k-1$ roots.

Informally, Theorems 2 and 4 say that best rational approximations of total degree n always reduce to polynomial approximations for functions of class Γ and never reduce to polynomial approximations for Stieltjes transforms. We observe that for $x^{1/2}$

$$R_{n,n}(x^{1/2}\colon [0, 1]) \leqslant e^{-c_1 n^{1/2}}$$

but

$$P_{2n}(x^{1/2}\colon [0, 1]) \geqslant c_2/n$$

and hence, that $R_{n,n}$ can be very much smaller than P_{2n} for functions satisfying the conditions of Theorem 2. (See [6, pp. 64 and 169].)

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